# Homology of strict $\omega$-categories

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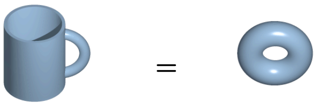
# De quoi parle cette thèse ?

Le thème sous-jacent est la théorie de l'homotopie des $\omega$-catégories strictes.

- Théorie de l'homotopie : étude des formes géométriques à déformation près



Outil principal en théorie de l'homotopie : les invariants homotopiques. C'est-à-dire qu'on associe à chaque forme géométrique une grandeur mathématique invariante par déformation.

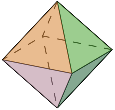
Exemple : nombre de composantes connexes (= nombre de morceaux).

Si deux formes géométriques n'ont pas le même nombre de morceaux, elles ne sont pas équivalentes à homotopie près.

# De quoi parle cette thèse ?

- Les $\omega$-catégories strictes sont des objets géométrico-algébriques.
  - Nature géométrique :

| $\omega$-catégorie stricte | polyèdre |
|---|---|
| 0-cellule | sommet |
| 1-cellule | arête |
| 2-cellule | face |
| $\ldots$ | $\ldots$ |



  - Nature algébrique : opérations sur les cellules.

En bref :

$\omega$-catégories strictes $\simeq$ façon algébrique de représenter des “formes géométriques”.

Ainsi, on peut faire la théorie de l'homotopie des $\omega$-catégories strictes. Dans cette thèse, on étudie et compare deux invariants sur les $\omega$-catégories strictes : l'un s'appelle l'homologie polygraphique et l'autre s'appelle l'homologie singulière...

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# Preliminary conventions

In this talk:

$\omega$-category = strict $\omega$-category

$n$-category = $\omega$-category with only unit cells above dimension $n$

1-category = (small) category

the functor $n\mathrm{Cat} \to \omega\mathrm{Cat}$ is an inclusion

# Orientals

Starting point: Street's *orientals*

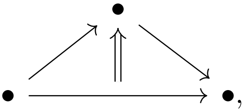
$$\mathcal{O}\colon \Delta \to \omega\mathsf{Cat}.$$

In pictures:

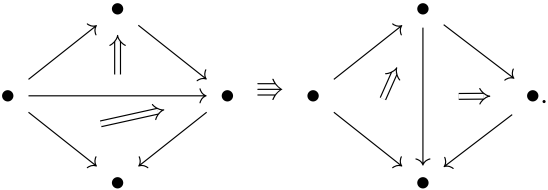
$$\mathcal{O}_0 = \bullet,$$

$$\mathcal{O}_1 = \bullet \longrightarrow \bullet,$$

$\mathcal{O}_2 =$



$\mathcal{O}_3 =$



# Nerve of $\omega$-categories

**Definition**

The nerve of an $\omega$-category $C$ is the simplicial set

$$\begin{aligned} N_\omega(C) : \Delta^{\mathrm{op}} &\to \mathsf{Set} \\ [n] &\mapsto \mathrm{Hom}_{\omega\mathsf{Cat}}(\mathcal{O}_n, C). \end{aligned}$$

This yields the nerve functor for $\omega$-categories

$$\begin{aligned} N_\omega : \omega\mathsf{Cat} &\to \widehat{\Delta} \\ C &\mapsto N_\omega(C). \end{aligned}$$

**Example**

When $C$ is a (1-)category, $N_\omega(C)$ is nothing but the usual nerve of $C$.

# Singular homology

Recall that to each simplicial set $X$, we can associate a chain complex

$$K(X) = \mathbb{Z}X_0 \longleftarrow \mathbb{Z}X_1 \longleftarrow \mathbb{Z}X_2 \longleftarrow \cdots$$

which allows to define the homology groups of $X$ as the homology groups of the chain complex $K(X)$.

**Definition**

Let $C$ be an $\omega$-category. The singular homology groups of $C$ are the homology groups of its nerve $N_\omega(C)$.

# Homotopy theory of $\omega$-categories

**Definition**

A morphism $f\colon C \to D$ of $\omega$Cat is a Thomason equivalence if $N_\omega(f)\colon N_\omega(C) \to N_\omega(D)$ is a weak equivalence of simplicial sets.

Let us write

$$\begin{aligned}\mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{Th}}) &:= \text{localization of } \omega\mathsf{Cat} \text{ w.r.t Thomason equivalences}\\ &= \text{“}\omega\text{-categories up to Thomason equivalences.”}\end{aligned}$$

$$\begin{aligned}\mathcal{H}\mathrm{o}(\widehat{\Delta}) &:= \text{localization of } \widehat{\Delta} \text{ w.r.t weak equivalences of simplicial sets}\\ &= \text{“simplicial sets up to weak equivalences.”}\end{aligned}$$

By definition, the nerve functor induces

$$\overline{N_\omega} : \mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{Th}}) \to \mathcal{H}\mathrm{o}(\widehat{\Delta}).$$

**Theorem (Gagna, 2018)**

$\overline{N_\omega} : \mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{Th}}) \to \mathcal{H}\mathrm{o}(\widehat{\Delta})$ is an equivalence of categories (or better an equivalence of derivators, or of weak $(\infty, 1)$-categories).

In other words:

Homotopy theory of $\omega$-categories induced by Thomason equivalences
$\cong$
Homotopy theory of spaces.

Hence,

Singular homology of $\omega$-categories
$\cong$
Homology of spaces.

# Polygraphs

## Definition

An $\omega$-category is free on a polygraph if it can be obtained recursively from the empty category by freely attaching cells.

Terminological convention:

$$\text{free } \omega\text{-category} = \omega\text{-category free on a polygraph.}$$

Example of free $\omega$-categories: the orientals.

## Important fact

If $C$ is a free $\omega$-category, then there is a *unique* set of generating cells possible.

# Polygraphic homology

Let $C$ be a free $\omega$-category and write $\Sigma_k$ for its set of generating $k$-cells.

**Definition**

The polygraphic homology of $C$ is the homology of the chain complex

$$\mathbb{Z}\Sigma_0 \xleftarrow{\partial} \mathbb{Z}\Sigma_1 \xleftarrow{\partial} \mathbb{Z}\Sigma_2 \xleftarrow{\partial} \cdots ,$$

where for $x \in \Sigma_n$, we have

$$\partial(x) = \text{“generators in the target of x”} - \text{“generators in the source of x”}.$$

Intuition?

Free $\omega$-categories $\cong$ CW-complexes
Polygraphic homology $\cong$ cellular homology

(Remark: Later we will see how to define polygraphic homology for all $\omega$-categories, not just free ones.)

# Polygraphic homology vs singular homology

A natural question:

Let $C$ be a (free) $\omega$-category. Do we have $H_\bullet^{\mathrm{pol}}(C) \simeq H_\bullet^{\mathrm{Sing}}(C)$ ?

Short answer: Not always. It depends on $C$.
(Hence, polygraphic homology doesn't work as well as cellular homology of CW-complexes.)

# Ara and Maltsiniotis' counter-example

Let $B$ be the 2-category freely generated by

- one object: $\bullet$,
- one 2-cell: $1_\bullet \Rightarrow 1_\bullet$.

$B =$ 

$B$ is free as an $\omega$-category and we have

$$H_k^{\mathrm{pol}}(B) \simeq \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

But (the nerve) of $B$ has the homotopy type of a $K(\mathbb{Z}, 2)$, hence $H_k^{\mathrm{Sing}}(B)$ is non-trivial for all even values of $k$.
Conclusion :

$$H_{2p}^{\mathrm{pol}}(B) \not\simeq H_{2p}^{\mathrm{Sing}}(B) \text{ for } p \geq 2.$$

However, as we shall see, there are tons of examples of $\omega$-categories for which singular homology and polygraphic homology do coincide.

**The fundamental question**

For which $\omega$-categories $C$ do we have $H_\bullet^{\mathrm{pol}}(C) \simeq H_\bullet^{\mathrm{Sing}}(C)$ ?

This is what I tried to answer in my PhD.

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# Abelianization of $\omega$-categories

Recall that by a variation of the Dold–Kan equivalence, we have:

$$\mathsf{Ab}(\omega\mathsf{Cat}) \simeq \mathsf{Ch}_{\geq 0},$$

where $\mathsf{Ch}_{\geq 0}$ is the category of non-negatively graded chain complexes. Hence, a forgetful functor

$$\mathsf{Ch}_{\geq 0} \simeq \mathsf{Ab}(\omega\mathsf{Cat}) \to \omega\mathsf{Cat},$$

which has a left adjoint

$$\lambda : \omega\mathsf{Cat} \to \mathsf{Ch}_{\geq 0},$$

referred to as the abelianization functor.

# Singular homology as a derived functor

$\mathcal{H}\mathrm{o}(\mathsf{Ch}_{\geq 0}) :=$ localization of $\mathsf{Ch}_{\geq 0}$ with respect to quasi-isomorphisms.

**Theorem (G. - 2020)**

The functor $\lambda \colon \omega\mathsf{Cat} \to \mathsf{Ch}_{\geq 0}$ is left derivable w.r.t the *Thomason equivalences* on $\omega\mathsf{Cat}$

$$\mathbb{L}\lambda^{\mathrm{Th}} \colon \mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{Th}}) \to \mathcal{H}\mathrm{o}(\mathsf{Ch}_{\geq 0}),$$

and for every $\omega$-category $C$ and every $k \geq 0$, we have

$$H_k^{\mathrm{Sing}}(C) = H_k(\mathbb{L}\lambda^{\mathrm{Th}}(C)).$$

From now on, we define the singular homology functor $\mathbb{H}^{\mathrm{Sing}}$ as

$$\mathbb{H}^{\mathrm{Sing}} := \mathbb{L}\lambda^{\mathrm{Th}} \colon \mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{Th}}) \to \mathcal{H}\mathrm{o}(\mathsf{Ch}_{\geq 0}).$$

# Equivalence of $\omega$-categories

**Definition**

An $\omega$-functor $f\colon C \to D$ is an equivalence of $\omega$-categories if :

- $f$ is essentially surjective (on 0-cells) up to a "reversible cell",
- for all 0-cells $x, y$ of $C$, the $\omega$-functor

$$\underline{\mathrm{Hom}}_C(x, y) \to \underline{\mathrm{Hom}}_D(f(x), f(y))$$

  is an equivalence of $\omega$-categories.

(Co-inductive definition.)

Example: When $C$ and $D$ are (1-)categories, we recover the usual notion of equivalence of categories.

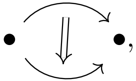
# Globes and spheres

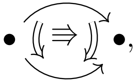
For every $n \in \mathbb{N}$,

- let $\mathbb{D}_n$ be the "$n$-globe" $\omega$-category:

$$\mathbb{D}_0 = \bullet,$$

$$\mathbb{D}_1 = \bullet \to \bullet,$$

$\mathbb{D}_2 =$ 

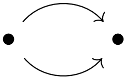
$\mathbb{D}_3 =$ 

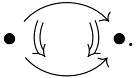
etc.

- let $\mathbb{S}_{n-1}$ be the "$(n-1)$-sphere" $\omega$-category:

$$\mathbb{S}_{-1} = \emptyset,$$

$$\mathbb{S}_0 = \bullet \qquad \bullet$$

$\mathbb{S}_1 =$ 

$\mathbb{S}_2 =$ 

etc.

- let $i_n : \mathbb{S}_{n-1} \to \mathbb{D}_n$ be the "boundary" inclusion.

# Equivalence of $\omega$-categories and the folk model structure

**Theorem (Lafont,Métayer,Worytkiewicz - 2010)**

There exists a model structure on $\omega$Cat such that:

- the weak equivalences are the equivalences of $\omega$-categories,
- the set $\{i_n : \mathbb{S}_{n-1} \to \mathbb{D}_n | n \in \mathbb{N}\}$ is a set of generating cofibrations.

It is known as the folk model structure on $\omega$Cat.

**Theorem (Métayer - 2008)**

The cofibrant objects of the folk model structure are exactly the free $\omega$-categories.

# Polygraphic homology as derived functor

**Proposition (folklore ?)**

The functor $\lambda : \omega\mathsf{Cat} \to \mathsf{Ch}_{\geq 0}$ is left Quillen w.r.t the folk model structure on $\omega\mathsf{Cat}$ and the projective model structure on $\mathsf{Ch}_{\geq 0}$.

In particular it is left derivable

$$\mathbb{L}\lambda^{\mathrm{folk}} \colon \mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{folk}}) \to \mathcal{H}\mathrm{o}(\mathsf{Ch}_{\geq 0}).$$

Moreover, for every free $\omega$-category $C$ and every $k \geq 0$, we have

$$H_k^{\mathrm{pol}}(C) \simeq H_k(\mathbb{L}\lambda^{\mathrm{folk}}(C)).$$

From now on, we define the polygraphic homology functor $\mathbb{H}^{\mathrm{pol}}$ as:

$$\mathbb{H}^{\mathrm{pol}} := \mathbb{L}\lambda^{\mathrm{folk}} \colon \mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{folk}}) \to \mathcal{H}\mathrm{o}(\mathsf{Ch}_{\geq 0}).$$

Conclusion: The polygraphic homology and the singular homology are obtained as left derived functors of the same functor, but not w.r.t to the same weak equivalences !

# Polygraphic homology for all

Note: We have extended the definition of polygraphic homology from free $\omega$-categories to *all* $\omega$-categories. When an $\omega$-category $C$ is not free, it suffices to a take cofibrant (= free) replacement of $C$.
For example:

**Proposition (Lafont,Métayer-2009)**

Let $M$ be a monoid (considered as an $\omega$-category). We have

$$H_\bullet^{\mathrm{pol}}(M) \simeq H_\bullet^{\mathrm{Sing}}(M).$$

Historically, this was the motivation for polygraphic homology.

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# Equivalence of $\omega$-categories vs Thomason equivalences

**Important Lemma**

Every equivalence of $\omega$-categories is a Thomason equivalence.

Consequence: the identity functor $\mathrm{id} : \omega\mathsf{Cat} \to \omega\mathsf{Cat}$ induces a functor

$$\mathcal{J} : \mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{folk}}) \to \mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{Th}}).$$

Remark: The converse of the above lemma is false. For example

$$\mathbb{D}_1 \to \mathbb{D}_0$$

is a Thomason equivalence but not an equivalence of $\omega$-categories.

# Canonical comparison map

**Proposition (abstract non-sense)**

There is a canonical natural transformation

$$\begin{array}{ccc} \mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{folk}}) & & \\ \mathcal{J}\downarrow & \overset{\pi}{\Rightarrow} \searrow^{\mathbb{H}^{\mathrm{pol}}} & \\ \mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{Th}}) & \xrightarrow[\mathbb{H}^{\mathrm{Sing}}]{} & \mathcal{H}\mathrm{o}(\mathsf{Ch}_{\geq 0}). \end{array}$$

In other words, for every $\omega$-category $C$ we have a map

$$\pi_C : \mathbb{H}^{\mathrm{Sing}}(C) \to \mathbb{H}^{\mathrm{folk}}(C),$$

which is natural in $C$. We refer to it as the canonical comparison map.

# Homologically coherent $\omega$-categories

**Definition**

An $\omega$-category $C$ is homologically coherent if the map

$$\pi_C : \mathbb{H}^{\mathrm{Sing}}(C) \to \mathbb{H}^{\mathrm{folk}}(C)$$

is an isomorphism.

Goal: Understand which $\omega$-categories are homologically coherent.

# Polygraphic homology is not homotopical

Another formal consequence of the formalism of left derived functors:

**Proposition (abstract non-sense)**

There exists at least one Thomason equivalence $u : C \to D$ such that the induced morphism

$$\mathbb{H}^{\mathrm{pol}}(C) \to \mathbb{H}^{\mathrm{pol}}(D)$$

is *not* an isomorphism.

In other words, if we think of $\omega$-categories as models for homotopy types, then the polygraphic homology is *not* a well-defined invariant!

**New slogan**

The polygraphic homology is a way of computing the singular homology of homologically coherent $\omega$-categories.

# Side note: equivalence of homologies in low dimension

**Proposition**

Let $C$ be *any* $\omega$-category. The canonical comparison map induces an isomorphism

$$H_k^{\mathrm{Sing}}(C) \to H_k^{\mathrm{pol}}(C)$$

for $k = 0, 1$.

For all $k \geq 4$, it is possible to find a $C$ such that

$$H_k^{\mathrm{pol}}(C) \not\simeq H_k^{\mathrm{Sing}}(C).$$

**Open question:**

Do we have

$$H_k^{\mathrm{pol}}(C) \simeq H_k^{\mathrm{Sing}}(C)$$

for $k = 2, 3$, for any $\omega$-category $C$ ?

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# Preliminaries: oplax contractile $\omega$-categories

**Definition**

An $\omega$-category $C$ is oplax contractible if the canonical morphism

$$C \to \mathbb{D}_0$$

has an inverse "up to an oplax transformation".

Example: A 1-category with a terminal object is oplax contractible.

**Lemma**

Every oplax contractible $\omega$-category is homologically coherent (and has the homotopy type of a point).

# An abstract criterion to detect homological coherence

Back to the triangle:

$$
\begin{array}{ccc}
\mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{folk}}) & & \\
\Big\downarrow \mathcal{J} & \overset{\pi}{\Rightarrow} \quad \searrow^{\mathbb{H}^{\mathrm{pol}}} & \\
\mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{Th}}) & \xrightarrow[\mathbb{H}^{\mathrm{Sing}}]{} & \mathcal{H}\mathrm{o}(\mathsf{Ch}_{\geq 0}).
\end{array}
$$

## Fundamental observation:

$\mathbb{H}^{\mathrm{pol}}$ and $\mathbb{H}^{\mathrm{Sing}}$ preserve homotopy colimits but $\mathcal{J}$ does *not* in general.

In other words, for a diagram $d : I \to \omega\mathsf{Cat}$, the canonical map

$$\underset{I}{\overset{\mathrm{Th}}{\operatorname{hocolim}}}(d) \to \underset{I}{\overset{\mathrm{folk}}{\operatorname{hocolim}}}(d)$$

is not an isomorphism in general.

Idea: exploit that sometimes it *is* an isomorphism.

# An abstract criterion to detect homological coherence

**Proposition**

Let $C$ be an $\omega$-category. Suppose that there exists $d : I \to \omega\mathsf{Cat}$ such that:

(i) $\underset{I}{\overset{\text{folk}}{\operatorname{hocolim}}}(d) \simeq \underset{I}{\overset{\text{Th}}{\operatorname{hocolim}}}(d) \simeq C$,

(ii) for each $i \in \mathrm{Ob}(I)$, the $\omega$-category $d(i)$ is homologically coherent.

Then $C$ is homologically coherent.

# Easy application: homology of globes and spheres

For every $n \geq 0$, $\mathbb{D}_n$ is oplax contractible, hence homologically coherent.Moreover, we have

$$\begin{array}{ccc} \mathbb{S}_{n-1} & \xrightarrow{i_n} & \mathbb{D}_n \\ {\scriptstyle i_n}\downarrow & \ulcorner & \downarrow \\ \mathbb{D}_n & \longrightarrow & \mathbb{S}_n, \end{array} \qquad (*)$$

(with $\mathbb{S}_{-1} = \emptyset$). This square is "folk homotopy cocartesian" because $i_n$ is a cofibration.

**Exceptional situation:**

The image by $N_\omega$ of $(*)$ in $\widehat{\Delta}$ is a *cocartesian* square of monos, hence homotopy cocartesian. It follows that square $(*)$ is "Thomason homotopy cocartesian".

By an immediate induction, $\mathbb{S}_n$ is homologically coherent (and has the homotopy type of an $n$-sphere).

# The case of 1-categories

**Theorem (G. - 2019)**

Every (small) category is homologically coherent.

Remark 1: The homology (polygraphic or singular) of a category need not be trivial above dimension 1.
Remark 2: Extension of Lafont and Métayer's result on the homology of monoids, but more precise and completely new proof.

# The case of 1-categories

*Sketch of proof:* Let $A$ be a small category. Recall that

$$\operatorname*{colim}_{a \in A} A/a \simeq A.$$

Moreover:

- each $A/a$ is oplax contractible, hence homologically coherent,
- $\operatorname*{hocolim^{Th}}_{a \in A} A/a \simeq \operatorname*{colim}_{a \in A} A/a \simeq A$ (From Thomason's homotopy colimit theorem).

The hard part is to show

$$\operatorname*{hocolim^{folk}}_{a \in A} A/a \simeq \operatorname*{colim}_{a \in A} A/a \simeq A.$$

Too long to explain but uses crucially the notion discrete Conduché $\omega$-functors (invented for this purpose).

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# 2-categories

We would like to understand which 2-categories are homologically coherent.

- For simplification, we focus on *free* 2-categories.
- Archetypal situation to understand: given a cocartesian square

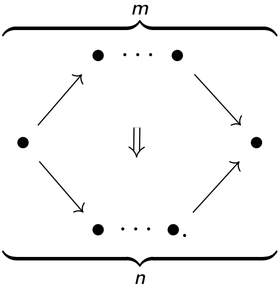
$$\begin{array}{ccc} \mathbb{S}_1 & \longrightarrow & P \\ {\scriptstyle i_1}\downarrow & \ulcorner & \downarrow \\ \mathbb{D}_2 & \longrightarrow & P' \end{array}$$

  with $P$ and $P'$ free 2-categories, when is it homotopy cocartesian w.r.t the Thomason equivalences ?
- I do not have a general answer to this question...
- However, using tools that I don't have time to explain, I know how to answer this question in many concrete situations.

# Zoology of 2-categories: basic examples

For $n, m \geq 0$, let $A_{(m,n)}$ be the free 2-category, with one generating 2-cell whose source is a chain of length $m$ and target a chain of length $n$:

$A_{(m,n)} =$




Examples:

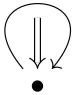
- $A_{(1,1)}$ is $\mathbb{D}_2$.
- $A_{(0,0)}$ is the 2-category $B$ from Ara and Maltsiniotis' counter-example.

# Zoology of 2-categories: basic examples

**Proposition**

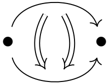
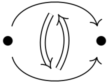
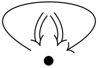
If $n + m > 0$, the 2-category $A_{(m,n)}$ has the homotopy type of a point and is homologically coherent.
Else, $A_{(0,0)}$ has the homotopy type of a $K(\mathbb{Z}, 2)$.

Remark: not that obvious for $m + n = 1$.
Example:

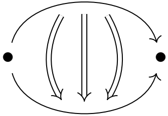
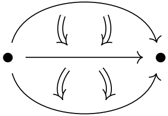
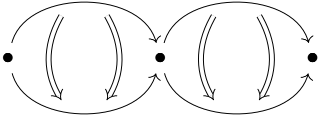
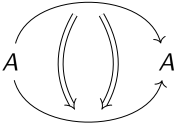
$A_{(1,0)} =$ 

has many non-trivial 2-cells.

# Zoology of 2-categories: variation of spheres

| 2-category | homologically coherent? | homotopy type |
|---|---|---|
|  | yes | $\mathbb{S}_2$ |
|  | yes | $\mathbb{S}_2$ |
|  | yes | $\mathbb{S}_2$ |
|  | yes | $\mathbb{S}_2$ |
|  | no | $K(\mathbb{Z}, 2)$ |
|  | no | $K(\mathbb{Z}, 2)$ |

# Zoology of 2-categories: Bouquets of spheres

| 2-category | homologically coherent? | homotopy type |
|---|---|---|
|  | yes | $\mathbb{S}_2 \vee \mathbb{S}_2$ |
|  | yes | $\mathbb{S}_2 \vee \mathbb{S}_2$ |
|  | yes | $\mathbb{S}_2 \vee \mathbb{S}_2$ |
|   | yes | $\mathbb{S}_2 \vee \mathbb{S}_1$ |

# Zoology of 2-categories: Torus

Let $C$ be the free 2-category pictured as

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & A \\ {\scriptstyle g}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle g} \\ A & \xrightarrow[\;f\;]{} & A. \end{array}$$

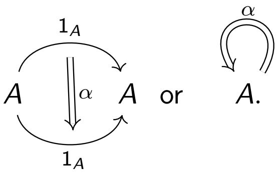
This 2-category has the homotopy type of the torus and is homologically coherent.

# Bubbles

## Definition

A bubble in a 2-category is a non unit 2-cell $\alpha$ whose source and target are units on a 0-cell.

In pictures:




## Definition

A 2-category is bubble-free if it has no bubbles.

# The bubble-free conjecture

The archetypal example of *non* bubble-free 2-category is the 2-category $B$ from Ara and Maltsiniotis' counter-example.
In all the examples, the free 2-categories that are homologically coherent are exactly the bubble-free ones.

**Conjecture**

Let $C$ be a free 2-category. It is homologically coherent if and only if it is bubble-free.

Merci pour votre attention !